REDDICK

# An Investigation of Green's Theorem for Discontinuous Functions

Graduate School

A. M.

1906

# AN INVESTIGATION OF GREEN'S THEOREM

# FOR

# DISCONTINUOUS FUNCTIONS

— BY —

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(A. B., Indiana University, 1904)

THESIS FOR THE DEGREE OF MASTER OF ARTS

IN THE GRADUATE SCHOOL

UNIVERSITY OF ILLINOIS

June, 1906

UNIVERSITY OF ILLINOIS

June 1st 190 6

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION * BY

Harry Wilfred Reddick

ENTITLED An Investigation of Green's Theorem for Discontinuous Functions.

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Master of Arts

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# Introduction

Green's Theorem first appeared in "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," which was published by subscription at Nottingham in 1828. The essay may be found in the "Mathematical Papers of the Late George Green" edited by N. M. Ferrers. The theorem has important applications in the theories of Electricity and Magnetism[1] Potential,[2] and Hydrodynamics.[3] The theorem expresses the surface integral of a function of two functions W and V and their partial derivatives with respect to the normal in terms of the volume integral of a function of the same two

1) Maxwell, Electricity and Magnetism, P.11 et seq.

2) Peirce, Newtonian Potential Function, P.32.

3) Lamb, Hydrodynamics, P 47 et seq.

functions W and V and their second differential parameters.

Suppose W and V and their first partial derivatives in any direction are uniform, continuous functions of the coordinates x, y, and z within a closed surface S and on the boundary. Also, let $D_n$ denote the partial derivative with respect to the exterior normal and let $\Delta$ denote the second differential parameter, namely, $\Delta = \frac{d}{dx^2} + \frac{d}{dy^2} + \frac{d}{dz^2}$. Then

$$\iint (W D_n V + V D_n W)\, dS = \iiint (W \Delta V - V \Delta W)\, dx\,dy\,dz, \quad (1)$$

where the triple integrals are to be extended to all points within S, and the surface integrals to all points on S.

For line and surface integrals this becomes

$$\int (W D_n V + V D_n W)\, ds = \iint (W \Delta V - V \Delta W)\, dx\,dy \quad (2)$$

where $u$ and $v$ and their derivatives are uniform continuous functions of $x$ and $y$ within a closed curve, $C$, the line integrals extending to all points on $C$ and the surface integrals to all points within $C$. $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$.

Equations (1) and (2) give Green's Theorem in the second form, in which it is the most often used. In the applications of the theorem, the integrands are assumed to be continuous. For example, if $u$ or $v$ represents the potential function due to a continuous distribution of matter, this function is everywhere finite.[1] Discontinuities in the function would have no physical application. Maxwell[2] has discussed the case where the functions

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1) Peirce, Newtonian Potential Function,

2) Maxwell, Electricity and Magnetism, P12.

$u$ or $V$ may not be uniform. It is the purpose of this thesis to investigate under what conditions different forms of this theorem hold when $u$ or $V$ becomes discontinuous at a set of points in the region considered.

# Line Integrals

1. <u>Definition of Line Integral</u>. — If a given line is divided in any way into infinitesimal elements, and the length of each element is multiplied by the value which a given point function, continuous along the line, has at some point within the element, the limit approached by the sum of these products as each element is indefinitely decreased, is called the <u>line integral</u> of the given function along the line in question.

2. <u>Complex Variable - Cauchy's Theorem</u> — In the theory of functions of a complex variable it is often necessary to consider the line integral of a function around a point in the complex plane at which the function has a singularity. Consider, for example, the

function $\frac{1}{z-a}$ which becomes infinite for $z=a$, i.e. has a pole at $a$. Integrating the function around a closed curve, C, enclosing the point $\underline{a}$, we have[1]

$$\int_C \frac{dz}{z-a} = 2\pi i \tag{3}$$

If $f(z)$ is a function of the complex variable $z$ which is regular within a region bounded by the closed curve, C, and if $\underline{a}$ is a point within this region, then we have the following more general theorem, the well known theorem of Cauchy:[2]

$$\int_C \frac{f(z)\,dz}{z-a} = 2\pi i\, f(a) \tag{4}$$

The coefficient of the (-1)st power of $z-a$ in the development of a function for the neighborhood of a pole $z=a$ is called the residue of the function for this pole. If $f(z)$ is expressed in the form:[3]

1) Burkhardt, Functionentheorie, V.1. p.114.

2) Ibid. p.115.

3) Ibid. p.137-138.

$$f(z) = \frac{A_{-n}}{(z-a)^n} + \frac{A_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + A_2(z-a)^2 + \cdots, \quad (5)$$

then
$$\int f(z)\,dz = 2\pi i\, A_{-1} \quad (6)$$

$A_{-1}$ being the residue of the function $f(z)$ relative to the pole $\underline{a}$. This is known as Cauchy's theorem of Residues.

The form (4) is a special case of (5) for, if $f(z)$ is developed into a series of the form
$$f(z) = A_{-1} + A_0(z-a) + A_1(z-a)^2 + \cdots \quad (7)$$
then
$$\frac{f(z)}{z-a} = \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + \cdots \quad (8)$$

and by (6) the integral of a function expanded in this form gives
$$\int \frac{f(z)\,dz}{z-a} = 2\pi i\, A_{-1} \quad (9)$$

But, by (7)
$$f(a) = A_{-1} \quad (10)$$

that is to say, the residue in this case is the value of the function at the point, and equation (9) is the same as equation (4).

3. <u>Real Variables — Condition that Line Integral is Independent of Path</u> — In equation (1) put $w = 1$; then

[illegible]

$$\int_C D_n V \, ds = \iint_A \Delta V \, dx\,dy \tag{11}$$

By this relation we will show that the condition that the line integral of $D_n V$ be independent of the path of integration, C, is equivalent to the the condition that V be a harmonic function everywhere within C. i.e. that V be uniform, continuous and satisfy the condition $\Delta V = 0$. For, if V is harmonic, we have

$$\Delta V = 0 \tag{12}$$

and, substituting this in the above equation, it reduces to

$$\int_C D_n V \, ds = 0 \tag{13}$$

where V is harmonic within C.

In the figure, the line integral of $D_n V\,ds$ around the shaded region following the arrows will be zero if V is harmonic between the curves C and c. For, by (13),

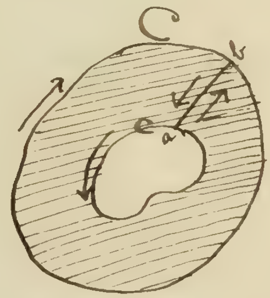


Fig. 1

$$\int_C D_n V\,ds + \int_b^a D_n V\,ds + \int_c D_n V\,ds + \int_a^b D_n V\,ds = 0 \tag{14}$$

But the integrals along $\underline{ab}$ in opposite directions cancel, and, reversing the direction of integration along c, we have

$$\int_C D_n V\,ds = \int_c D_n V\,ds \tag{15}$$

i.e. the integral is independent of the path.

## 4. Cauchy's Theorem for Real Variables —

Consider the line integral of the function $\frac{1}{\sqrt{(x-a)^2+(y-b)^2}}$ around a circle K with center $(a,b)$. We have

$$\int_K \frac{ds}{\sqrt{(x-a)^2+(y-b)^2}} = \int_0^{2\pi} \frac{r\,d\theta}{r} = 2\pi \tag{16}$$

the value of the integral is independent of the size of the circle. Also, since the function $\frac{1}{r}$ is harmonic, by (15) the integral will be the same around any closed curve enclosing the point $(a,b)$. So we may write:

$$\int_C \frac{ds}{\sqrt{(x-a)^2+(y-b)^2}} = 2\pi \tag{17}$$

where C is any closed curve surrounding the point $(a,b)$.

Consider the line integral of the function

$$\frac{f(xy)}{\sqrt{(x-a)^2+(y-b)^2}} \tag{18}$$

around any closed curve, C, enclosing the point $(a,b)$, where $f(xy)$ is continuous everywhere within C. The function (18) becomes infinite at one point $(a,b)$ within the contour C. Surround the point $(a,b)$ by a small circle K with arbitrarily small radius, $r_K$. Then we may apply (15) provided the function (18) is of the form $D_n V$ i.e. the further condition must be imposed on the function $f(xy)$ that it is of the form $r D_n V$, where V is a harmonic function in the region between C and K. Then

$$\int_C \frac{f(xy)\,ds}{\sqrt{(x-a)^2+(y-b)^2}} = \int_K \frac{f(xy)\,ds}{\sqrt{(x-a)^2+(y-b)^2}} \tag{19}$$

This second integral may be written

$$\int_K \frac{f(xy)\,ds}{\sqrt{(x-a)^2+(y-b)^2}} = \int_K \frac{f(ab)\,ds}{\sqrt{(x-a)^2+(y-b)^2}} + \int_K \frac{f(xy)-f(ab)}{\sqrt{(x-a)^2+(y-b)^2}}\,ds \quad (20)$$

By (16), $\displaystyle\int_K \frac{f(ab)\,ds}{\sqrt{(x-a)^2+(y-b)^2}} = f(ab)\int \frac{ds}{\sqrt{(x-a)^2+(y-b)^2}} = 2\pi f(ab),$ (21)

Also, by virtue of the continuity of $f(x,y)$, for an arbitrarily small $\epsilon$ we can find a $\delta$ such that

$$|f(x,y)-f(a,b)| < \epsilon \quad (22)$$

when $|x-a|<\delta \quad |y-b|<\delta \quad$ i.e. $r_K < \delta\sqrt{2}$

The third integral may be written

$$\int_K \frac{[f(x,y)-f(a,b)]\,r_K\,d\theta}{r_K} \quad (23)$$

The $r_K$'s will always cancel, and, by making $r_K$ small enough, we can make $|f(x,y)-f(a,b)|<\epsilon$. Hence this integral vanishes we thus have

$$\int_C \frac{f(xy)\,ds}{\sqrt{(x-a)^2+(y-b)^2}} = 2\pi f(ab) \quad (24)$$

This theorem is analagous to Cauchy's Theorem for a complex variables, (4). The

extension to the case where there is a set of poles within the contour will be made later.

5. Residue Theorem for a Single Pole — We will now develop a theorem similar to (24) in another form, using the theory by which the theorems in the next section, for three dimensions, are deduced. This theory for three variables has been developed by Appell[1]

Let $V_n(x,y)$ denote the most general homogeneous polynomial of degree $n$ in $x$ and $y$ satisfying the equation

$$\Delta V_n = \frac{\partial^2 V_n}{\partial x^2} + \frac{\partial^2 V_n}{\partial y^2} = 0 \qquad (25)$$

Consider a function $F(xy)$, uniform in all space and regular at all points except certain points which we call singular points. If $P(a,b)$ is one of these points, we say that this point is a pole or non-essential singular point of

1) Acta Mathematica, V III-IV P 313-334

degree $n$ of the function F if there exists a function $\phi$, of the form

$$\phi(x-a,y-b) = V_{-1}(x-a,y-b)+V_{-2}(x-a,y-b)+\cdots+V_{-n}(x-a,y-b) \qquad (26)$$

such that

$$F(xy) - \phi(x-a,y-b) \qquad (27)$$

is regular at the point P. The first term of $\phi$ is of the form

$$V_{-1}(x-a,y-b) = \frac{\lambda}{\sqrt{(x-a)^2+(y-b)^2}} \qquad (28)$$

and the coefficient $\lambda$ is called the <u>residue</u> of the function F relative to the pole P. If no such function $\phi$ exists, P is called an <u>essential</u> <u>singular</u> <u>point</u>.

A function F(xy), uniform, continuous and having partial first and second derivatives and satisfying $\Delta F = 0$ at all points within the space between two circles with the same center (a, b) and radii R, R'; R > R', is developable into a series of the form

$$F(x,y) = \sum_{\nu=-\infty}^{\nu=+\infty} V_\nu(x-a, y-b) \qquad (29)$$

uniformly convergent at all points between the two circles. This is analagous to Laurent's theorem for complex variables.

We will now prove the following

Theorem — If a function $F(xy)$, satisfying the equation $\Delta F = 0$, has a pole or essential singular point $P$, the line integral

$$\int_C D_n F \, ds$$

around any closed contour $C$, enclosing the point $P$, and containing no other singular points, is equal to $-2\pi$ times the residue of $F$ relative to the point $P$.

Let $K$ be any circle with center $(a,b)$ and radius $r$ lying wholly within $C$; then, since the function $F$ is harmonic in the region between $C$ and $K$, we have, by applying (15),

$$\int_C D_n F \, ds = \int_K D_r F \, ds \qquad (30)$$

that is to say, the determination of the integral around the curve C may be replaced by that around a circle K. Between this circle K and an arbitrarily small concentric circle within K the function F can be developed by (29) in the form

$$F(x,y) = \sum_{\nu=-\infty}^{\nu=\infty} V_\nu(x-a, y-b) \qquad (31)$$

On the circle K,

$$x - a = r\cos\theta$$
$$y - b = r\sin\theta \qquad 0 \leqq \theta \leqq 2\pi$$

Substituting in (31)

$$F(x,y) = \sum_{\nu=-\infty}^{\nu=+\infty} V_\nu[r\cos\theta, r\sin\theta]$$

But $V_\nu(r\cos\theta, r\sin\theta) = r^\nu Y_\nu(\theta)$, where $Y_\nu$ denotes a function of LaPlace. Then

$$F(xy) = \sum_{\nu=1}^{\nu=\infty} r^\nu Y_\nu(\theta) + Y_0(\theta) + \sum_{\nu=1}^{\nu=\infty} r^{-\nu} Y^{(1)}_{\nu-1}(\theta) \qquad (32)$$

where $Y_0(\theta)$ is a constant and $Y^{(1)}$ denotes another $Y$ function with different constants.

then:

$$D_r F = \sum_{\nu=1}^{\nu=\infty} \nu r^{\nu-1} Y_\nu(\theta) - \sum_{\nu=1}^{\nu=\infty} \nu r^{-\nu-1} Y^{(1)}_{\nu-1}(\theta) \qquad (33)$$

Now substituting in $\int_K D_r F\, ds$ the value of $D_r F$ in (33) and also $ds = r\,d\theta$. Also note that for any function of Laplace, $Y_\nu$,

$$\int_0^{2\pi} Y_\nu(\theta)\, d\theta = 0 \quad \text{for } \nu \neq 0 \qquad (34)$$

since $Y_\nu(\theta)$ is homogeneous in $\sin\theta$ and $\cos\theta$.

We then have

$$\int_K D_r F\, ds = -\int_0^{2\pi} Y_0^{(1)}(\theta)\, d\theta = -2\pi Y_0^{(1)}, \qquad (35)$$

$Y_0^{(1)}$ being the residue of F relative to the point P.

## 6. Extension of Residue Theorem to n Poles.

Suppose now that the function $F(xy)$ has any finite number of poles or essential singular points within the contour C. We will demonstrate for this case the following:

Theorem. — If a function $F(xy)$, satisfying the equation $\Delta F = 0$, is uniform

and regular at all points within and on a contour, C, except at certain interior points $P_1 P_2 \cdots P_p$, the line integral

$$\int_C D_n F\, ds$$

taken around the contour, C, is equal to $-2\pi$ times the sum of the residues of the function F relative to these points $P_1 P_2 \cdots P_p$.

Surround each point $P_i$ by a circle $K_i$ situated within C and having for its center the point $P_i$ and a radius small enough so that it will not cut any of the other circles $K_j$ having for centers the points $P_j$. Representing by A the area interior to C and exterior to the circles $K_1 K_2 \cdots K_p$, the function F is regular throughout this area and, by (13), the line integral

$$\int D_n F\, ds = 0$$

taken along the contour limiting the area A. But this integral is the sum of

$p+1$ integrals taken along the contours $C, K_1, K_2, \ldots K_p$, so that

$$\int_C D_n F\, ds + \sum_{k=1}^{k=p} \int_{K_k} D_n F\, ds = 0 \tag{36}$$

By the preceding theorem

$$\int_{K_k} D_n F\, ds = 2\pi R_k \quad , \tag{37}$$

$R_k$ denoting the residue of $F$ relative to the point $P_k$, the + sign being used since the derivative is taken with respect to the exterior normal to the contour enclosing the area $A$, i.e. the interior normal to the circle $K_k$. Then, from (36)

$$\int_C D_n F\, ds = -2\pi\,(R_1 + R_2 + \cdots R_p) \quad , \tag{38}$$

which proves the theorem.

# Surface Integrals

7. Definition of Surface Integral — If a given surface is divided in any way into infinitesimal elements such that the distance between the two most widely separated points within each element is infinitesimal and the area of each element is multiplied by the value which a given point function, continuous over the surface, has at some point within the element, the limit approached by the sum of these products as each element is indefinitely decreased, is called the surface integral of the given function over the surface in question.

8. Case Where the Function becomes Infinite or Indeterminate[1] — Under what conditions does $\iint f(xy)\,dx\,dy$ have a meaning if $f(xy)$ becomes infinite or indeterminate within the

1) Picard, Traité d'Analyse. V.1. Chap VII.

field of integration?

Suppose $(a,b)$ to be the singular point. Trace around the point a small closed curve $\gamma$ and extend the integral to that part of the domain without $\gamma$. If this integral approaches a limit always the same as $\gamma$ approaches indefinitely near the point $(a,b)$ by any law whatever, then this limit represents the value of the integral. Consider the integral

$$\int_0^a\int_0^{a'} \frac{\partial^2 V}{\partial x\,\partial y}\,dx\,dy \tag{39}$$

over the rectangle formed by the coordinate axes and the lines $x=a$, $y=a'$. First suppose $V$ continuous within and on the rectangle of integration. Then, integrating with respect to $x$, the integral (39) becomes

$$\int_0^{a'} \frac{\partial}{\partial y}\left[V(a,y)-V(0,y)\right]dy \tag{40}$$

$$= V(a,a') - V(a,0) - V(0,a') + V(0,0) \tag{41}$$

Now suppose that $V(x,y)$ is infinite or indeterminate for $x=0$, $y=0$. Take for $\gamma$ a

small rectangle with sides $\epsilon\epsilon'$ parallel to the axes. ~~Denoting~~ the rectangles so formed by I, II, III, we have

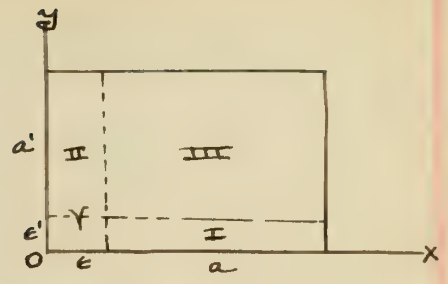

Fig 2.

$$\int_{\epsilon}^{a}\int_{0}^{\epsilon'}\frac{\partial^2 V}{\partial x\,\partial y}\,dx\,dy = V(a,\epsilon') - V(a,0) - V(\epsilon,\epsilon') + V(\epsilon,0) \qquad (42)$$
I

$$\int_{0}^{\epsilon}\int_{\epsilon'}^{a'}\frac{\partial^2 V}{\partial x\,\partial y}\,dx\,dy = V(\epsilon,a') - V(\epsilon,\epsilon') - V(0,a') + V(0,\epsilon') \qquad (43)$$
II

$$\int_{\epsilon}^{a}\int_{\epsilon'}^{a'}\frac{\partial^2 V}{\partial x\,\partial y}\,dx\,dy = V(a,a') - V(a,\epsilon') - V(\epsilon,a') + V(\epsilon,\epsilon') \qquad (44)$$

Subtracting (44) from the sum of (42) and (43), we have

$$\iint_{I+II+III}\frac{\partial^2 V}{\partial x\,\partial y}\,dx\,dy = V(a,a') - V(a,0) - V(0,a') + V(\epsilon,0) + V(0,\epsilon') - V(\epsilon,\epsilon') \qquad (45)$$

If (45) has a definite value, independent of the limit of the ratio $\frac{\epsilon}{\epsilon'}$, it defines the integral. The last three terms of (45) may, however, become infinite or have a limit depending on the limit of the ratio $\frac{\epsilon}{\epsilon'}$, in which case the integral has no meaning.

Example:

$$V = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial^2 V}{\partial x\,\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}\,dx\,dy$$

$$\iint\limits_{I+II+III} \frac{y^2-x^2}{(x^2+y^2)^2}\,dx\,dy = \tan^{-1}\frac{a'}{a} - \tan^{-1}\frac{0}{a} - \tan^{-1}\frac{a'}{0} + \tan^{-1}\frac{0}{\epsilon} + \tan^{-1}\frac{\epsilon'}{0} - \tan^{-1}\frac{\epsilon'}{\epsilon}$$

$$= \tan^{-1}\frac{a'}{a} - \tan^{-1}\frac{\epsilon'}{\epsilon}$$

Since this latter expression depends on the limit of $\frac{\epsilon'}{\epsilon}$, the integral has no meaning extended to the rectangle $(a, a')$.

An analagous method may be employed in the case of triple integrals. Surround the singular point by a sphere and study the limit of the integral as this sphere approaches zero. If the function becomes infinite at a set of isolated points, we may surround each point by a sphere and study the limit of the integral as these spheres together approach zero.

We thus have a criterion for determining whether the integrals have a meaning when singular points occur in the field of integration.

9. Theorems on Harmonic Functions of Three Real Variables — If, in the second form of Green's Theorem for surface and

volume integrals, equation (1), we put $V=1$ and $w = F$ where $\Delta F = 0$, we have

$$\iint D_n F \, ds = 0$$

and hence we may state the theorem analagous to (13):

Theorem I — If a function $F(x,y,z)$, satisfying the equation $\Delta F = 0$, is uniform and regular at all points within and on a surface $S$, then

$$\iint_S D_n F \, ds = 0$$

Appell[1] has proved the following theorems for three variables by a method analagous to that used in Art. 5 for functions of two variables.

Theorem II[2] — If a function $F(x,y,z)$, satisfying

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1) P. Appell, "Sur Les Fonctions de Trois Variables Réeles Satisfaisant à l'Equation Differentielle $\Delta F = 0$" – Acta Math. V III–IV.

2) Since the integral of a function F over any ~~surface~~ is the same as the integral over a sphere if the function satisfies the equation $\Delta F = 0$ between the surface and the sphere, this theorem will hold when the integral is extended to any surface enclosing the point P.

the equation $\Delta F = 0$, has a pole or essential singular point $P$, the integral

$$\iint D_n F\, ds$$

extended to the surface of a sphere with center $P$ and containing no other singular points, is equal to $-4\pi$ times the residue of $F$ relative to the point $P$.

Theorem III — If a function, $F(x,y,z)$, satisfying the equation $\Delta F = 0$, is uniform within a surface $S$ and regular at all points within and on $S$ except at certain interior points $P_1 P_2 \cdots P_p$, the integral

$$\iint D_n F\, ds$$

extended to the surface $S$ is equal to $-4\pi$ times the sum of the residues relative to the points $P_1 P_2 \cdots P_p$.

This theorem is proved by enclosing each point $P_k$ by a sphere $S_k$ and proceeding by a method similar to that of art. 5.

Theorem IV — If $F(x,y,z)$ is a function satisfying the equation $\Delta F = 0$, uniform within a closed surface $S$ and regular at all points within and on $S$, then

$$F(a,b,c) = \frac{1}{4\pi}\iint_S (F D_n T - T D_n F)\, dS \qquad (46)$$

where $(a,b,c)$ is a fixed point within $S$ and

$$T = \frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

$n$ referring to the exterior normal. For the proof of this theorem see Heine, Kugelfunctionen. Vol 2. P. 93. As an example, let $F$ be the Newtonian Potential Function, $V$, due to any distribution of matter, and $T = \frac{1}{r}$, where $r$ is the distance from a point $P$ within the space included by $S$. Then we will have[1]

$$V_P = \frac{1}{4\pi}\iint_S \left(V D_n\left(\tfrac{1}{r}\right) - \tfrac{1}{r} D_n V\right) dS \qquad (47)$$

where $V_P$ is the value of the potential at the

[1] Webster, Electricity and Magnetism. P. 164.

point P. i.e. the value of a harmonic function at any point is determined if the value of the function together with that of its normal component is given at all points of a surface S.

We have similarly[1], if V represents the logarithmic potential function

$$V_P = \frac{1}{2\pi}\int_C (\log r\, D_n V - V\, D_n \log r)\, ds \quad (48)$$

We thus see the analogy between these theorems and Cauchy's Theorem for complex variables, viz

$$f(s) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-s}\, dz \quad (49)$$

which gives the value of a regular function of a complex variable, $z$, at any point $s$ within a closed curve $\Gamma$ in terms of the value of the same function on the boundary $\Gamma$.

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1) Webster, Elec. + Mag. P. 179.

# Curvilinear Integrals

10. Definition of Curvilinear Integral —[1] Let $P(x,y)$ be a point function of $x$ and $y$, and $C$ a curve in the $xy$ plane joining the points $\alpha$ and $\beta$, with coordinates $(a,A)$ and $(b,B)$. Divide the curve $C$ into

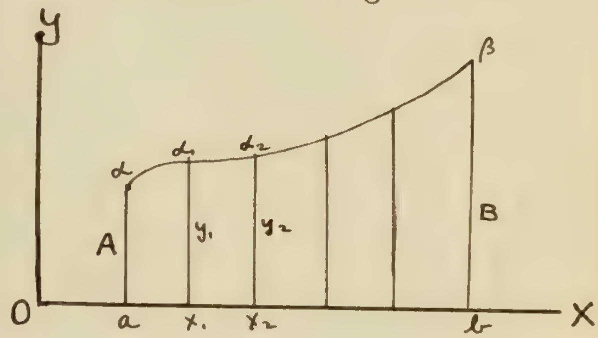


Fig. 3.

any number of intervals by the points whose coordinates are $(x_1 y_1)$ $(x_2 y_2)$ ---- $(x_{n-1} y_{n-1})$. Then form the sum:

$$P(a,A)(x_1-a)+P(x_1 y_1)(x_2-x_1)+----+P(x_{n-1} y_{n-1})(b-x_{n-1})$$

which differs from that of the ordinary definite integral only in that the function $P$ depends on $y$ as well as on $x$.

We then take the limit of this sum as the intervals $(\alpha_i, \alpha_{i+1})$ approach zero, their number increasing indefinitely.

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1) Picard, Traité d'Analyse. V.1, P70-71.

Let $y = \phi(x)$ be the equation of the curve $\alpha\beta$, where $\phi(x)$ is single-valued and continuous from $a$ to $b$. If we put $P[x, \phi(x)] = F(x)$, the sum (50) takes the form

$$F(a)(x_1-a) + F(x_1)(x_2-x_1) + \cdots + F(x_{n-1})(b-x_{n-1}), \qquad (51)$$

the limit of which is the definite integral

$$\int_a^b P(x, \phi(x))\, dx$$

This integral, represented by the symbol

$$\int_C P(xy)\,dx \qquad (52)$$

we call the <u>curvilinear</u> <u>integral</u> of $P(xy)$ along the curve $C$ from $\alpha$ to $\beta$. Its value depends on the path from $\alpha$ to $\beta$ as well as on the end points.

We define likewise the curvilinear integral

$$\int_C Q(xy)\,dy \qquad (53)$$

as the limit of the sum

$$Q(a,A)(y_1-A) + Q(x_1y_1)(y_2-y_1) + \cdots + Q(x_{n-1}y_{n-1})(\beta - y_{n-1}), \qquad (54)$$

the summation being taken in this case with respect to $y$.

11. <u>Green's Theorem</u> — In equations (1) and (2), Green's Theorem in its second form was stated for two and three dimensional regions. A special case of (2), when $W=1$ and $V$ harmonic, was deduced in Art. 3. In Art. 9. a special case of (1) was given, where $V=1$ and $W=F$, a harmonic function. Other examples of (1) and (2), when one function is harmonic and the other the Newtonian or logarithmic potential function, were given in Art 9, e.g. We will now develop Green's Theorem for curvilinear integrals.[1)]

Suppose $C$ is a closed curve enclosing an area $A$, and $P(x,y)$ and $Q(x,y)$ are given point functions of $x$ and $y$. We will consider the

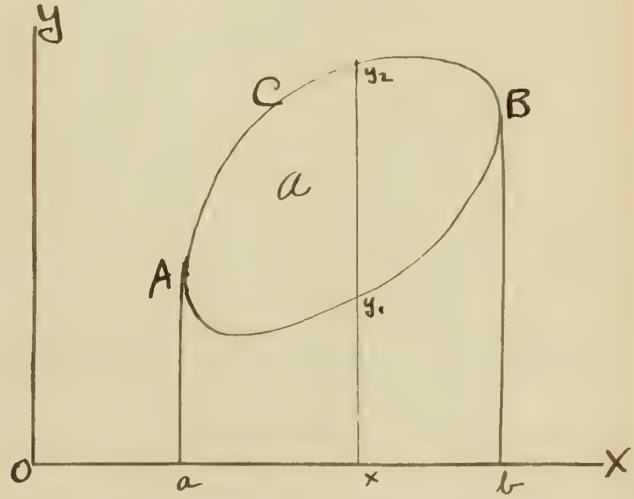


Fig. 4.

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Maschke, Lecture Notes on Function Theory, University of Chicago 1905.

curvilinear integral

$$\int_C P\,dx + Q\,dy \tag{55}$$

obtained by combining the integrals (52) and (53)

Let $P_1$ represent the function $P(xy)$ along the lower part of the curve $C$ from $B$ to $A$ i.e. $P_1 = P(xy_1)$ and let $P_2$ represent the function $P(xy)$ along the upper part of the curve $C$ from $A$ to $B$ i.e. $P_2 = P(xy_2)$, $y_1$ and $y_2$ being the ordinates of the lower and upper points of the curve respectively. We will integrate around $C$ in the negative direction i.e. clockwise.

$$\begin{aligned}\int_C P\,dx &= \int_a^b P_2\,dx + \int_b^a P_1\,dx \\ &= \int_a^b P_2\,dx - \int_a^b P_1\,dx \\ &= \int_a^b (P_2 - P_1)\,dx \end{aligned} \tag{56}$$

We will find the surface integral $\iint_a D_y P\,dx\,dy$ by first integrating along a strip parallel to the $y$-axis, $y$ varying from $y_1$ to $y_2$ and then summing these strips.

$$\iint_a D_y P\,dx\,dy = \int_a^b \Big[\int_{y_1}^{y_2} D_y P\,dy\Big]\,dx \tag{57}$$

Then, if $P$ is continuous from $y_1$ to $y_2$, i.e. continuous everywhere within $C$,

$$\int_{y_1}^{y_2} D_y P\,dy = P\Big]_{y_1}^{y_2} = P_2 - P_1$$

Hence
$$\iint_a D_y P\,dx\,dy = \int_a^b (P_2 - P_1)\,dx \qquad (58)$$

and so, by equation (56), we have

$$\int_C P\,dx = \iint_a D_y P\,dx\,dy \qquad (59)$$

Consider now $\int_C Q\,dy$. Let $Q_1$ represent the function $Q(x,y)$ along the left part of the curve $C$ from $D$ to $E$ i.e. $Q_1 = Q(x_1, y)$, and let $Q_2$ represent the function $Q(x,y)$ along

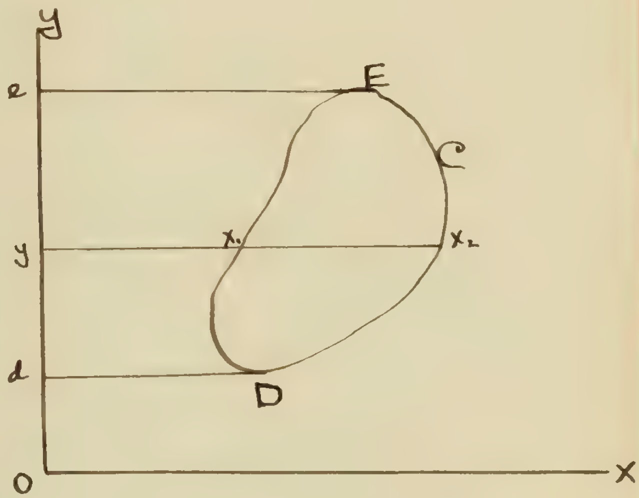


Fig. 5.

the right part of the curve $C$ from $E$ to $D$. i.e. $Q_2 = Q(x_2, y)$, $x_1$ and $x_2$ being the abscissas of the left and right points of the curve respectively. Integrating around $C$ in the negative direction, we have

$$\int_C Q\,dy = \int_d^e Q_1\,dy + \int_e^d Q_2\,dy$$
$$= \int_d^e (Q_1 - Q_2)\,dy \qquad (60)$$

We will find the surface integral $\iint\limits_{a} D_x Q\, dx\, dy$ by first integrating along a strip parallel to the x-axis, x varying from $x_1$ to $x_2$ and then summing these strips. We thus have

$$\iint\limits_{a} D_x Q\, dx\, dy = \int_{d}^{e}\Big[\int_{x_1}^{x_2} D_x Q\, dx\Big] dy \qquad (61)$$

But, if Q is not discontinuous within C,

$$\int_{x_1}^{x_2} D_x Q\, dx = Q\Big]_{x_1}^{x_2} = Q_2 - Q_1$$

Hence

$$\iint\limits_{a} D_x Q\, dx\, dy = \int_{d}^{e} (Q_2 - Q_1)\, dy \qquad (62)$$

and so, by equation (60), we have

$$\int\limits_{C} Q\, dy = -\iint\limits_{a} D_x Q\, dx\, dy \qquad (63)$$

Combining (59) and (63) we have Green's Theorem; viz,

$$\int\limits_{C} P\,dx + Q\,dy = \iint\limits_{a} (D_y P - D_x Q)\, dx\, dy \qquad (64)$$

in which the curvilinear integral on the left is taken in the negative direction and P and Q are point functions of x and y

which are continuous throughout the region.

12. Condition that Integral shall be Independent of Path. — The necessary and sufficient condition that $\int_C P\,dx + Q\,dy$ be independent of the path C is deduced in Picard, Traité d'Analyse, V.1. p 73-79, and is shown to be

$$D_yP = D_xQ$$

If in equation (64) we put $D_yP = D_xQ$ we have

$$\int_C P\,dx + Q\,dy = 0$$

which agrees with the above result.

We have shown in Art. 3 that if V is harmonic within C we have

$$\int_C D_nV\,ds = 0$$

we will now show that the relation

$$\int_C P\,dx + Q\,dy = 0 \quad , \quad D_yP = D_xQ \, , \tag{65}$$

for curvilinear integrals, is equivalent to the relation

$$\int_C D_nV\,ds = 0 \quad , \quad \Delta V = 0 \, , \tag{66}$$

for line integrals.

Since $$D_n V = D_x V \, D_n x + D_y V \, D_n y$$

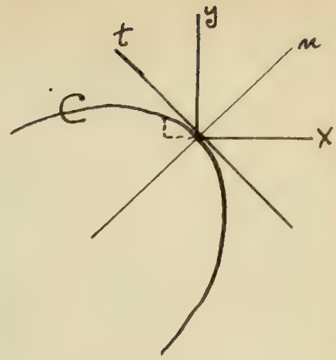


Fig 6

we have

$$D_n V \, ds = D_x V \cos(xn)\, ds + D_y V \cos(yn)\, ds$$

where $\cos(xn)$ denotes the cos. of the angle between $x$ and $n$, etc, Fig 6.

But $$\cos(xn) = \cos(yt)$$

and $$\cos(yn) = \cos(xt)$$

Then $$\cos(xn)\, ds = \cos(yt)\, ds = dy$$

and $$\cos(yn)\, ds = \cos(xt)\, ds = -dx$$

Hence $$D_n V \, ds = D_x V \, dy - D_y V \, dx$$

Putting $P = -D_y V$ and $Q = D_x V$, this becomes

$$D_n V \, ds = P\,dx + Q\,dy$$

Then (65) is equivalent to (66), the condition

$$D_y P = D_x Q$$

becoming $$-D_y^2 V = D_x^2 V$$

or $$\Delta V = 0$$

As was stated before, equation (64), and hence condition (65) does not hold if $P$ or $Q$ becomes infinite within the contour $C$.

As an example we will take

$$P = -\frac{y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2}$$

Here P and Q satisfy the condition $D_yP = D_xQ$

for.
$$D_yP = \frac{y^2-x^2}{(x^2+y^2)^2} = D_xQ$$

But P and Q both become infinite at the point (0, 0). For by substituting

$$x = r\cos\theta$$
$$y = r\sin\theta$$

we have $P = -\frac{\sin\theta}{r}$ and $Q = \frac{\cos\theta}{r}$

both of which become infinite for $r = 0$,

In this case

$$\int P\,dx + Q\,dy,$$

taken around a circle with center at the origin, becomes

$$\int_C \frac{x\,dy - y\,dx}{x^2+y^2} = \int_C \frac{r^2\,d\theta}{r^2}$$

$$= \int_0^{2\pi} d\theta = 2\pi \qquad (65)$$

# Proper and Improper Integrals

13. Definition of Proper Integral [1] We shall denote by an ordinary line any closed or open line which has in common with parallels to the $x$ and $y$ axes a finite number of points. Let $F$ be a region in the $xy$ plane bounded by ordinary lines and let $f(xy)$ be uniquely defined for every point within this region and on the boundary. Let this region $F$ be divided into polygons by a network of ordinary lines. For convenience let the lines be chosen parallel to the $x$ and $y$ axes, in which case $F$ will be divided into rectangles $r_1 r_2$ ------ $r_n$, some of them reaching over the boundary of $F$. In each rectangle $r_r$ there must be points of $F$. and each

[1] Stolz, Gründzuge der Differential und Integralrechnung, V.3. P37 etseq.

point of $F$ must belong to some rectangle $\tau_r$. The number of rectangles, $n$, is arbitrary. Let $g_r$ and $k_r$ be the upper and lower limits respectively of $f(xy)$ over the rectangle $\tau_r$. Then

$$\lim_{\tau_r \doteq 0} \sum_1^n g_r \tau_r = G \qquad (66)$$

is called the upper double integral of the function $f(x,y)$ over the region $F$, and is denoted by Jordan as

$$S'_F f(xy)\, dA \qquad (67)$$

$dA$ representing the area of the arbitrarily small rectangles $\tau_r$. Also

$$\lim_{\tau_r = 0} \sum_1^n k_r \tau_r = K \qquad (68)$$

is called the lower double integral of the function $f(x,y)$ over the region $F$ and is denoted by

$$S^2_F f(x,y)\, dA \qquad (69)$$

where $S$ is a symbol for the summation

extended to the area $\mathfrak{F}$ for an arbitrary mode of subdivision.

If $G = K$, then their common value is called the Proper Double Integral of the function $f(x,y)$ over the region $\mathfrak{F}$ and is denoted by

$$S_{\mathfrak{F}}\, f(x,y)\, dA. \qquad (70)$$

or, in the case where the polygons $T_r$ are rectangles, $dA$ becomes $dx\,dy$ and the above expression may be written

$$\iint_{\mathfrak{F}} f(xy)\, dx\, dy \qquad (71)$$

We would get the same limits $G$ and $K$ by using only those rectangles which lie wholly within $\mathfrak{F}$ or reach to the boundary of $\mathfrak{F}$. We will make use of the following theorems[1]:

Theorem I — A real function $f(x,y)$ which at every point of a region $\mathfrak{F}$, the

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1) Stolz, vol 3, P. 69-70.

boundary included, is continuous with respect to both variables $x$ and $y$, has a proper double integral over this region.

Theorem II – A function $f(x, y)$ which is finite throughout a region $F$ and continuous with respect to $x$ and $y$ at all points of $F$, with the exception of a finite number of isolated points and points on a finite number of ordinary lines, has a proper integral over this region. The value of the integral is independent of the values of the function at these points of discontinuity, but the discontinuities must be finite.

14. Green's Theorem for Proper Integrals – The previous statements of this theorem have been for a single closed boundary. We may generalize the theorem by extending it to a boundary of $n$ rims. Let $F$ be a region bounded by an outer

rim $r$ and $m$ inner rims, $r_1, r_2 --- r_m$. The special case for one rim is obtained by putting $m = 0$. Suppose that these rims are traced in a positive sense by a point whose coordinates $x, y$ are expressed as functions of an increasing parameter $\tau$. Denote the coordinates of this point for the outer rim by

$$x_\tau = \phi(\tau), \quad y_\tau = \psi(\tau) \qquad \alpha \leqq \tau \leqq \beta$$

and for the inner rim by

$$x_{r\tau} = \phi_r(\tau), \quad y_{r\tau} = \psi_r(\tau), \quad \alpha_r \leqq \tau \leqq \beta_r, \; r = 1, 2 -- m.$$

The intervals $(\alpha, \beta)$ $(\alpha_r \beta_r)$ may be either finite or infinite. The functions $\phi(\tau), \psi(\tau)$, $\phi_r(\tau), \psi_r(\tau)$ are continuous for every value of $\tau$ in the interval, end points included, and have continuous differential quotients at all points, except, perhaps, a finite number. The function $f(xy)$ has a proper double integral by Theorem II; hence we can write:

$$\iint_{F} f(x,y)\, dx\, dy = J \qquad (72)$$

Theorem III [1] Let $F(x,y)$ be a continuous function at all points of the domain $\mathfrak{F}$, whose partial derivative with respect to $x$ is $D_xF$, at least at all points of $\mathfrak{F}$ where $D_xF$ is continuous; then,

$$\iint_{\mathfrak{F}} D_xF\,dx\,dy = \int_{\alpha}^{\beta} F(x_\tau, y_\tau)\frac{dy_\tau}{d\tau}\,d\tau - \sum_{1}^{m}{}_{r} \int_{\alpha_r}^{\beta_r} F(x_{r,\tau}\, y_{r,\tau})\frac{dy_{r,\tau}}{d\tau}\,d\tau \quad (73)$$

Theorem IV – Let $G(x,y)$ be a continuous function at all points of the domain $\mathfrak{F}$, whose partial derivative with respect to $y$ is $D_yG$ at least at all points of $\mathfrak{F}$ where $D_yG$ is continuous; then

$$\iint_{\mathfrak{F}} D_yG\,dx\,dy = -\int_{\alpha}^{\beta} G(x_\tau y_\tau)\frac{dx_\tau}{d\tau}\,d\tau + \sum_{1}^{m}{}_{r} \int_{\alpha_r}^{\beta_r} G(x_{r,\tau}\, y_{r,\tau})\frac{dx_{r,\tau}}{d\tau}\,d\tau \quad (74)$$

Combining equations (73) and (74), we get

Theorem V –

$$\iint_{\mathfrak{F}} (D_xF - D_yG)\,dx\,dy =$$

$$\int_{\alpha}^{\beta} F(x_\tau y_\tau)\frac{dy_\tau}{d\tau}\,d\tau + G(x_\tau y_\tau)\frac{dx_\tau}{d\tau}\,d\tau$$

$$- \sum_{1}^{m}{}_{r} \int_{\alpha_r}^{\beta_r} F(x_{r,\tau} y_{r,\tau})\frac{dy_{r,\tau}}{d\tau} + G(x_{r,\tau} y_{r,\tau})\frac{dx_{r,\tau}}{d\tau}\,d\tau \quad (75)$$

[1] For proof of Theorems III and IV, see Stolz, V. 3, P. 97 et seq.

which is the second form of Green's theorem for proper integrals for $m+1$ rims. This is an extension of (64) and reduces to that form if $m=0$ and we replace the F and G functions by Q and P respectively. The sign, however, will be changed, since in (75) the rims are traced in the positive direction. We note further that if $D_xF = D_yG$ in (75), which is the condition that the integral be independent of the path, we will have the curvilinear integral around the $m+1$ rims equal to zero or the integral around the outer rim, $r$, equal to that around the $m$ inner rims, $r_1, r_2 ---- r_m$.

15. <u>Definition</u> <u>of</u> <u>Improper</u> <u>Integral</u> —

Suppose that the function $f(x,y)$ becomes infinite at one or more points of a finite domain $\mathfrak{F}$, limited by ordinary lines.

However, let $f(x,y)$ be finite at least within every region $G$ which lies wholly within $F$, i.e., has no points in common with the boundary of $F$. If a region should contain points in whose neighborhood the function is infinite, we may draw lines through these points and thus divide the region into smaller regions having the property of $F$.

Under these conditions the function $f(x,y)$ does not possess a proper integral over the region $F$, but it will possess a proper integral over any region $G$ having an ordinary boundary and lying wholly within $F$. Denote it by

$$S_G\, f(x,y)\, dA$$

If this integral has a limit always the same as $G$ approaches $F$ in any way, we denote this limit by

$$J = S_{F} f(x,y)\, dA \tag{76}$$

and call it the <u>improper double integral</u> of $f(x,y)$ over the region $F$.

That is, for an arbitrarily small $\epsilon > 0$ a $\delta > 0$ exists such that

$$\left| J - S_{G} f(x,y)\, dA \right| < \epsilon$$

for

$$0 < F - G < \delta$$

Let

$$x = \phi(\tau, \sigma), \quad y = \psi(\tau, \sigma),$$

represent the coordinates of a point tracing the curve or curves which approach the boundary of $F$ as the parameter $\sigma$ approaches $0$. Denoting by $G(\sigma)$ the region bounded by these variable curves, we have

$$\lim_{\sigma = +0} G(\sigma) = F$$

We may then state the following

<u>Theorem</u> — If the function $f(x,y)$ has, over the region $F$, an improper

integral

$$J = S_{F} f(x,y)\, dA$$

then $J$ is the limit of the proper integral of $f(x,y)$ over the variable region $G(\sigma)$ which approaches $F$ indefinitely, i.e.

$$J = \lim_{\sigma=+0} S_{G(\sigma)} f(x,y)\, dA \qquad (77)$$

## 16. Green's Theorem for Improper Integrals.

Suppose that the following conditions hold. (1) The boundary of a region $F$ is composed of regular lines, i.e. if

$$x = \phi(\tau),\ y = \psi(\tau),\quad \alpha \leqq \tau \leqq \beta \qquad (78)$$

are the equations of the outer rim $r$, then the functions $\phi(\tau)$ and $\psi(\tau)$ are finite and have continuous derivatives with respect to $\tau$ for each $\tau$ in the interval $(\alpha,\beta)$ or $r$ is divided into a finite number of parts in each of which $\phi(\tau)$ and $\psi(\tau)$ possess this property.

(2) The functions $D_xF$ and $D_yG$ satisfy the conditions of art. 14 within a region which lies wholly within $\mathfrak{F}$. (3) The functions $F$ and $G$ are defined as in Theorems III and IV, art. 14. (4) The functions $D_xF$ and $D_yG$ possess an improper integral over the region $\mathfrak{F}$.

Take a system of regular curves $r_\sigma$ whose equations are

$$x = x_{\tau\sigma} = \phi(\tau,\sigma)\ , \quad y = y_{\tau\sigma} = \psi(\tau,\sigma)\ , \quad (\alpha \leqq \tau \leqq \beta) \quad (79)$$

and which approach the outer rim $r$ as $\sigma$ approaches 0. Also take a system of regular curves $r_{1\sigma}$ whose equations are

$$x = x_{1\tau\sigma} = \phi_1(\tau,\sigma)\ , \quad y = y_{1\tau\sigma} = \psi_1(\tau,\sigma)\ , \quad (\alpha_1 \leqq \tau \leqq \beta_1) \quad (80)$$

and which approach the inner rim $r_1$ as $\sigma$ approaches 0. The intervals $(\alpha,\beta)$ $(\alpha_1,\beta_1)$ are finite. We may take

$\sigma$ so small that the curves (79) and (80) do not cut, in which case they enclose a region which we will call $G(\sigma)$. Then, in this region, the functions $D_xF$ and $D_yG$ are properly integrable and we have by equations (73) and (74)

$$\mathcal{S}_{G(\sigma)} D_xF\,dx\,dy = \int_{\alpha}^{\beta} F(x_{\tau\sigma}\,y_{\tau\sigma})\frac{dy_{\tau\sigma}}{d\tau}\,d\tau - \int_{\alpha_1}^{\beta_1} F(x_{1\tau\sigma}\,y_{1\tau\sigma})\frac{dy_{1\tau\sigma}}{d\tau}\,d\tau \quad (81)$$

and

$$\mathcal{S}_{G(\sigma)} D_yG\,dx\,dy = -\int_{\alpha}^{\beta} G(x_{\tau\sigma}\,y_{\tau\sigma})\frac{dx_{\tau\sigma}}{d\tau}\,d\tau + \int_{\alpha_1}^{\beta_1} G(x_{1\tau\sigma}\,y_{1\tau\sigma})\frac{dx_{1\tau\sigma}}{d\tau}\,d\tau \quad (82)$$

Now, since $D_xF$ and $D_yG$ are improperly integrable over $\mathfrak{F}$, then, by (77)

$$J_1 = \lim_{\sigma=+0} \mathcal{S}_{G(\sigma)} D_xF\,dx\,dy$$

and

$$J_2 = \lim_{\sigma=+0} \mathcal{S}_{G(\sigma)} D_yG\,dx\,dy$$

Taking the limits of both members of equations (81) and (82) as $\sigma$ approaches 0, we get

$$J_1 = \iint_{\mathfrak{F}} D_x F\, dx\, dy = \int_{\alpha}^{\beta} F(x_\tau y_\tau) \frac{dy_\tau}{d\tau}\, d\tau - \int_{\alpha_1}^{\beta_1} F(x_{1\tau} y_{1\tau}) \frac{dy_{1\tau}}{d\tau}\, d\tau \quad (83)$$

and

$$J_2 = \iint_{\mathfrak{F}} D_y G\, dx\, dy = -\int_{\alpha}^{\beta} G(x_\tau y_\tau) \frac{dx_\tau}{d\tau}\, d\tau + \int_{\alpha_1}^{\beta_1} G(x_{1\tau} y_{1\tau}) \frac{dx_{1\tau}}{d\tau}\, d\tau \quad (84)$$

Combining equations (83) and (84), we get

$$\iint_{\mathfrak{F}} (D_x F - D_y G)\, dx\, dy =$$

$$\int_{\alpha}^{\beta} F(x_\tau y_\tau) \frac{dy_\tau}{d\tau}\, d\tau + G(x_\tau y_\tau) \frac{dx_\tau}{d\tau}\, d\tau$$

$$- \int_{\alpha_1}^{\beta_1} F(x_{1\tau} y_{1\tau}) \frac{dy_{1\tau}}{d\tau}\, d\tau + G(x_{1\tau} y_{1\tau}) \frac{dx_{1\tau}}{d\tau}\, d\tau \quad (85)$$

which is the second form of Green's theorem for improper integrals.

In equation (75) the functions $D_x F$ and $D_y G$ may have at most only finite discontinuities at a finite number of isolated points or at points on a finite number of ordinary lines. In this latter

form of the theorem (85), the functions $D_xF$ and $D_yG$ may have infinite discontinuities at points along a finite number of ordinary lines. The curvilinear integrals in the right members of these equations (75) and (85) must, however, exist if the theorems hold. Hence the set of ~~points~~ of discontinuity which lies on the rims must be such that the curvilinear integrals along the rims exist.

## 17. Special Case of the Preceding Theorem—

The following theorem was proved by M. B. Porter in the Annals of Math. V.7 No.1 Oct. 1905.

Let $R$ be a region bounded by an ordinary closed curve $C$ and lying between the lines $y = a$ and $y = b$. If $F(x,y)$ and $D_xF$ be defined for all those points within $R$ lying on the lines $y = y_k$, where the points $y_k$ are everywhere dense in $(a,b)$, and if, $F(x,y)$ being defined for all points of $C$, the integrals $\int_C F(xy)\,dy$ and $\iint_R D_xF(x,y)\,dx\,dy$ exist in Riemann's sense[1], then

$$\int_C F(xy)\,dy = \iint_R D_xF\,dx\,dy \qquad (86)$$

We may make this extension: Let the region $R$ also lie between the lines $x = c$ and $x = d$; then, if $G(x,y)$ and $D_yG$ be defined

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1) i.e. The functions $F(x,y)$ and $D_xF$ are either properly or improperly integrable.

for all those points within R lying on the lines $x = x_K$, where the points $x_K$ are everywhere dense in $(c, d)$, and if, $G(x,y)$ being defined for all points of C, the integrals $\int_C G(x,y)\,dx$ and $\iint_R D_y G\,dx\,dy$ exist in Riemann's sense, then

$$-\int_C G(x,y)\,dx = \iint_R D_y G\,dx\,dy \qquad (87)$$

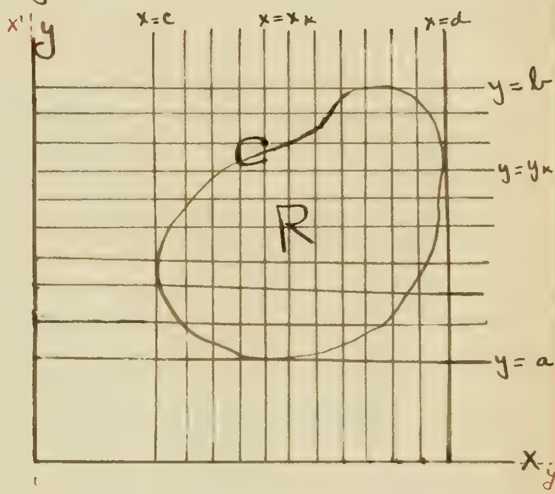

Fig. 8.

We may show this as follows. Turn the axes to the right through an angle of 90°. Denote by x' the direction opposite to x. Then the old x axis is replaced by the y axis and the old y axis by the x' axis. Now, forming an integral similar to (86) for a G function, we have

$$\int_C G(y,-x')\,dx' = \iint_R D_y G(y,-x')\,dx'\,dy = J \qquad (88)$$

Since the element of the axis is changed in sign while the element of area is always positive, i.e.

the improper integral $J$ will not change, but the curvilinear integral changes sign, we obtain (87) by substituting in (88)

$$x' = -x$$
$$dx' = -dx$$
$$dx'dy = dxdy$$

Combining equations (86) and (87), we get

$$\iint_R (D_x F - D_y G)\,dx\,dy = \int_C F(x,y)\,dy + G(x,y)\,dx \qquad (89)$$

which shows that Green's Theorem holds under the given conditions, where the functions $D_x F$ and $D_y G$ may become infinite at point sets lying on lines $y = \bar{y}_K$ and $x = \bar{x}_K$ respectively, where $\bar{y}_K$ and $\bar{x}_K$ represent the sets complementary to $y_K$ and $x_K$ respectively. The point sets over which these functions become infinite are of content zero.

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